

S-duality of D-brane action at order $O(\alpha'^2)$

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Abstract

Using the compatibility of the DBI and the Chern-Simons actions with the T-duality transformations, the curvature corrections to these actions have been recently extended to include the quadratic B-field couplings at order $O(\alpha'^2)$. In this paper, we use the compatibility of the couplings on D₃-brane with the S-duality to find the nonlinear RR couplings at order $O(\alpha'^2)$. We confirm the quadratic RR couplings in the DBI part with the disk-level scattering amplitude. Using the modular functions that appear in the curvature corrections, we then write the results in $SL(2, Z)$ invariant form.

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1 Introduction

The low energy effective field theory of D-branes consists of the Dirac-Born-Infeld (DBI) [1] and the Chern-Simons (CS) actions [2]. The curvature corrections to the CS part can be found by requiring that the chiral anomaly on the world volume of intersecting D-branes (I-brane) cancels with the anomalous variation of the CS action. This action for a single D_p -brane at order $O(\alpha'^2)$ is given by [3, 4, 5],

$$S_{CS} \supset -\frac{\pi^2 \alpha'^2 T_p}{24} \int_{M^{p+1}} C^{(p-3)} \wedge \left[\text{tr}(R_T \wedge R_T) - \text{tr}(R_N \wedge R_N) \right] \quad (1)$$

where M^{p+1} represents the world volume of the D_p -brane. For totally-geodesic embeddings of world-volume in the ambient spacetime, $R_{T,N}$ are the pulled back curvature 2-forms of the tangent and normal bundles respectively (see the appendix in ref. [6] for more details).

The curvature corrections to the DBI action has been found in [6] by requiring consistency of the effective action with the $O(\alpha'^2)$ terms of the corresponding disk-level scattering amplitude [7, 8]. For totally-geodesic embeddings of world-volume in the ambient spacetime, the corrections in the string frame for zero B-field and for constant dilaton are [6]

$$S_{DBI} \supset \frac{\pi^2 \alpha'^2 T_p}{48} \int d^{p+1}x e^{-\phi} \sqrt{-G} \left[R_{abcd} R^{abcd} - 2\hat{R}_{ab} \hat{R}^{ab} - R_{abij} R^{abij} + 2\hat{R}_{ij} \hat{R}^{ij} \right] \quad (2)$$

where $\hat{R}_{ab} = G^{cd} R_{cadb}$ and $\hat{R}_{ij} = G^{cd} R_{cidj}$. Here also a tensor with the world-volume or transverse space indices is the pulled back of the corresponding bulk tensor onto world-volume or transverse space². For the case of D_3 -brane with trivial normal bundle the curvature couplings (1) and (2) have been modified in [6] to include the complete sum of D-instanton corrections by requirement of the $SL(2, Z)$ invariance of the couplings.

In the presence of non-constant dilaton, the couplings (2) are not consistent with T-duality. For zero B-field, the compatibility with linear T-duality requires the following extension of (2):

$$S_{DBI} \supset \frac{\pi^2 \alpha'^2 T_p}{48} \int d^{p+1}x e^{-\phi} \sqrt{-G} \left[R_{abcd} R^{abcd} - 2(\hat{R}_{ab} - \phi_{,ab})(\hat{R}^{ab} - \phi^{,ab}) \right. \\ \left. - R_{abij} R^{abij} + 2(\hat{R}_{ij} - \phi_{,ij})(\hat{R}^{ij} - \phi^{,ij}) \right] \quad (3)$$

where commas denote partial differentiation³. The dilaton couplings for D_3 -brane, however, breaks the S-duality of the curvature terms found in [6]. Using the fact that the dilaton and

²Our index conversion is that the Greek letters (μ, ν, \dots) are the indices of the space-time coordinates, the Latin letters (a, d, c, \dots) are the world-volume indices and the letters (i, j, k, \dots) are the normal bundle indices.

³Using on-shell relations, the definition of the curvature tensor $\hat{R}_{\mu\nu}$ in [9] has been changed as $\hat{R}_{\mu\nu} \equiv \frac{1}{2}(R_{\mu a}{}^a{}_{\nu} - R_{\mu k}{}^k{}_{\nu})$. With this tensor the couplings in (2) are then invariant under linear T-duality [9] when B-field is zero. If one uses the standard definition $\hat{R}_{\mu\nu} \equiv R_{\mu a}{}^a{}_{\nu}$, then the couplings (3) are invariant.

the RR zero-form transform similarly under the S-duality transformations, one expects there should be similar couplings as above for the RR zero-form. We will show that the S-matrix element of two RR vertex operators found in [7] produces in fact such couplings. Having similar couplings for the dilaton and the RR zero-form, we use an appropriate $SL(2, Z)$ matrix to write the results in $SL(2, R)$ invariant form. However, there is an overall factor of $e^{-\phi}$ in the Einstein frame which is not invariant under the $SL(2, R)$ transformation.

In the presence of non-zero B-field, the couplings (1) and (3) are not consistent with the T-duality. Using the compatibility of these couplings with linear T-duality as a guiding principle, the quadratic B-field couplings at order $O(\alpha'^2)$ have been found in [9, 10, 11]. The B-field couplings in the DBI part are [9]⁴

$$S_{DBI} \supset -\frac{\pi^2 \alpha'^2 T_p}{48} \int d^{p+1} x e^{-\phi} \sqrt{-G} \left[\frac{1}{6} H_{ijk,a} H^{ijk,a} + \frac{1}{3} H_{abc,i} H^{abc,i} - \frac{1}{2} H_{bci,a} H^{bci,a} \right] \quad (4)$$

The above couplings have been confirmed with the disk level S-matrix calculations [9]. These couplings for D₃-brane again break the S-duality of the curvature terms found in [6]. Using the fact that the B-field and the RR two-form appear as a doublet in the S-duality transformations, one expects there should be similar couplings as above for the RR two-form. We will show that the S-matrix element of two RR vertex operators found in [7] produces in fact such couplings. Having similar couplings for the B-field and the RR two-form, we then write the results in $SL(2, R)$ invariant form. In this case also, there is an overall factor of $e^{-\phi}$ in the Einstein frame which is not invariant under the $SL(2, R)$ transformation.

It has been shown in [13] that the CS part should include couplings which involve linear NSNS field. These couplings have been found by studying the S-matrix element of one RR and one NSNS vertex operators at order $O(\alpha'^2)$ [7]. These couplings in string frame are [13]⁵

$$S_{CS} \supset -\frac{\pi^2 \alpha'^2 T_p}{24} \int d^{p+1} x \epsilon^{a_0 \dots a_p} \left(\frac{1}{2!(p-1)!} [F_{ia_2 \dots a_p, a}^{(p)} H_{a_0 a_1}{}^{a, i} - F_{aa_2 \dots a_p, i}^{(p)} H_{a_0 a_1}{}^{i, a}] \right) \quad (5)$$

⁴In this paper, we are interested only in the quadratic order of field strengths, *e.g.*, we are not considering H^4 , RH^2 , or $R\partial\phi\partial\phi$ terms in DBI action.

⁵Using on-shell relations, the definition of the curvature tensor \hat{R}_{ij} in [13] has been changed as $\hat{R}_{ij} \equiv \frac{1}{2}(R_{ia}{}^a{}_j - R_{ik}{}^k{}_j)$. With this tensor the coupling $F_{a_0 \dots a_p j, i}^{(p+2)} \hat{R}^{ij}$ is then invariant under linear T-duality [13]. If one uses the standard definition $\hat{R}_{ij} \equiv R_{ia}{}^a{}_j$, then the second term in the second line in (5) can be written at the linear order as $F_{a_0 \dots a_p}^{(p+2) j, i} (h_{ij, aa} + h_{aa, ij} - h_{ia, aj} - h_{ja, ai} - 2\phi_{, ij})/2(p+1)$ where h is the metric perturbation. Under T-duality along the world volume direction y , the RR factor $F_{a_0 \dots a_p}^{(p+2) j, i}/(p+1)$ which includes the Killing index y , transforms to $F_{a_0 \dots a_{p-1}}^{(p+1) j, i}$. This RR field, however, does not include the Killing index. Hence, the indices i, j in the T-dual theory do not include the Killing index y . Using this observation, one can easily verify that the metric/dilaton factor $(h_{ij, aa} + h_{aa, ij} - h_{ia, aj} - h_{ja, ai} - 2\phi_{, ij})$ is invariant under the linear T-duality. Hence, the second term in the second line in (5) is invariant under the T-duality.

$$\begin{aligned}
& + \frac{2}{p!} \left[\frac{1}{2!} F_{ia_1 \dots a_p j, a}^{(p+2)} R^a{}_{a_0}{}^{ij} - \frac{1}{p+1} F_{a_0 \dots a_p j, i}^{(p+2)} (\hat{R}^{ij} - \phi^{,ij}) \right] \\
& - \frac{1}{3!(p+1)!} F_{ia_0 \dots a_p j k, a}^{(p+4)} H^{ijk, a} \Big)
\end{aligned}$$

The redundant fields $F^{(6)}, \dots, F^{(9)}$ in above action are related to the magnetic dual of the RR field strengths $F^{(1)}, \dots, F^{(4)}$ as $F^{(10-n)} = *F^{(n)}$ for $n = 1, 2, 3, 4$. For the self-dual D₃-brane in the Einstein frame, we will show that up to the overall factor of $e^{-\phi}$, the above couplings can be written in $SL(2, Z)$ invariant forms.

The S-duality of the above couplings is like the S-duality of the R^4 corrections to the supergravity action [14, 15, 16, 17]. In the Einstein frame these couplings have an overall factor of $e^{-3\phi/2}$. It has been conjectured in [18] that this factor is in fact the leading order term of the non-holomorphic Eisenstein series E_s with $s = 3/2$ at weak coupling. This conjecture has been confirmed with one loop [18] and two loops [19]. We speculate that the above $O(\alpha'^2)$ corrections to D₃-brane action can be written in $SL(2, Z)$ invariant form by extending the weak coupling factor $e^{-\phi}$ to the regularized non-holomorphic Eisenstein series E_s with $s = 1$. This function appears also in the R^2 terms of the D₃-brane action [6, 20].

The quadratic B-field couplings have been added to the CS action (1) by requiring the consistency of this action with the linear T-duality [10, 11]. These couplings are

$$S_{CS} \supset \frac{-\pi^2 \alpha'^2 T_p}{24 \times 2!2!(p-3)!} \int d^{p+1} x \epsilon^{a_0 \dots a_p} C_{a_4 \dots a_{p-4}}^{(p-3)} \left[\frac{1}{2} H_{a_0 a_1 a, i} H_{a_2 a_3}{}^{a, i} - \frac{1}{2} H_{a_0 a_1 i, a} H_{a_2 a_3}{}^{i, a} \right] \quad (6)$$

They have been confirmed by the S-matrix element of one RR and two NSNS vertex operators [10, 12]. For the D₃-brane case, S-duality indicates that there should be similar couplings for $CC^{(2)}C^{(2)}$. Including these couplings one can write them in $SL(2, Z)$ invariant form. However, in this case the overall factor is the axion field which may be written in the S-dual form by replacing it with another $SL(2, Z)$ invariant function $f(\tau, \bar{\tau})$ whose weak-expansion starts at axion instead of the dilaton in the Eisenstein series.

An outline of the paper is as follows: We begin the section 2 by examining the disk-level S-matrix element of two RR vertex operators from which we find the quadratic RR couplings on the world volume of D_p-branes at order $O(\alpha'^2)$. In section 3, we show that for the self-dual D₃-brane case and for the RR two-form, the couplings are exactly the same as the B-field couplings (4). We then write the result in $SL(2, Z)$ invariant form using the $SL(2, R)$ matrix \mathcal{M} which appears in the Type IIB supergravity action, and the regularized non-holomorphic Eisenstein series E_s with $s = 1$. In this section we write also the couplings of the RR four-form and the RR scalar in S-dual form using $E_1(\tau, \bar{\tau})$. In section 4, we write the CS couplings in $SL(2, Z)$ invariant form. We end this paper by summarizing the new disk level couplings that are predicted by requiring the consistency of the DBI action (2) and the CS action (1) with the S-duality.

2 RR couplings from S-matrix

The scattering amplitude of two RR states from D_p -brane is given by [7]

$$\begin{aligned} A(\varepsilon_1, p_1; \varepsilon_2, p_2) &= -\frac{\alpha'^2 T_p}{16 \times 32} K(1, 2) \frac{\Gamma(-\alpha' t/4) \Gamma(\alpha' q^2)}{\Gamma(1 - \alpha' t/4 + \alpha' q^2)} \\ &= \frac{T_p}{16 \times 32} K(1, 2) \left(\frac{4}{q^2 t} + \zeta(2) \alpha'^2 + O(\alpha'^4) \right) \end{aligned} \quad (7)$$

where $q^2 = p_1^a p_1^b \eta_{ab}$ is the momentum flowing along the world-volume of D-brane, and $t = -(p_1 + p_2)^2$ is the momentum transfer in the transverse direction. The kinematic factor is

$$K(1, 2) = \left(2q^2 a_1 + \frac{t}{2} a_2 \right) \quad (8)$$

where

$$a_1(n, m, p) = -\frac{1}{2} \text{Tr}(P_- \Gamma_{1(n)} M_p \gamma_\mu C^{-1} M_p^T \Gamma_{2(m)}^T C \gamma^\mu) \quad (9)$$

$$a_2(n, m, p) = \frac{1}{2} \text{Tr}(P_- \Gamma_{1(n)} M_p \gamma_\mu) \text{Tr}(P_- \Gamma_{2(m)} M_p \gamma^\mu)$$

where

$$\begin{aligned} \Gamma_{\alpha(n)} &= \frac{1}{n!} (F_\alpha)_{\nu_1 \dots \nu_n} \gamma^{\nu_1} \dots \gamma^{\nu_n} \\ M_p &= \frac{\pm 1}{(p+1)!} \epsilon_{a_0 \dots a_p} \gamma^{a_0} \dots \gamma^{a_p} \end{aligned} \quad (10)$$

where F_α for $\alpha = 1, 2$ is the linearized RR field strength n -form and ϵ is the volume $p+1$ -form of the D_p -brane. In equation (8), $P_- = \frac{1}{2}(1 - \gamma_{11})$ is the chiral projection operator. The γ_{11} gives the magnetic couplings and 1 gives the electric couplings. The first term in (7) produces the massless poles resulting from the $(\alpha')^0$ order of the DBI and CS couplings on the D-brane, and the supergravity couplings in the bulk. The second term in (7) should produce $(\alpha')^2$ couplings of two RR fields on the world volume of D_p -brane in which we are interested.

Using various identities, a_1 can be simplified for electric components of the RR field strength to [7]

$$a_1(n, m, p) = \frac{8}{n!} \delta_{mn} \left[\text{Tr}(D) F_{1(n)} \cdot F_{2(n)} - 2n D^\lambda{}_\kappa F_{1\lambda\nu_2 \dots \nu_n} F_2^{\kappa\nu_2 \dots \nu_n} \right] \quad (11)$$

where the matrix $D^\mu{}_\nu$ is diagonal with +1 in the world volume directions and -1 in the transverse directions. The degree of the RR field strength n in a_1 is independent of the

dimension of the D_p -brane. For the magnetic components, one finds the same result but for different n , *i.e.*, $n' = 10 - n$. Using the fact that the redundant field strength $F^{(10-n)}$ for $n = 1, 2, 3, 4$ are the magnetic dual of $F^{(n)}$, *i.e.*, $F^{(10-n)} = *F^{(n)}$, one finds the following result for the magnetic components of the RR vertex operator:

$$a_1(n', m', p) = \frac{8}{n'!} \delta_{m'n'} \left[\text{Tr}(D)(*F_{1(n)}) \cdot (*F_{2(n)}) - 2n' D^\lambda (*F^{(n)})_{1\lambda\nu_2\cdots\nu_{n'}} (*F^{(n)})_2^{\kappa\nu_2\cdots\nu_{n'}} \right]$$

The degree of the RR field strength in a_2 depends on the dimension of the D_p -brane. To see this consider the factor $\text{Tr}(P_- \Gamma_{1(n)} M_p \gamma_\mu)$ in a_2 . For the electric components it is nonzero only for $n = p$ and for $n = p + 2$, and for magnetic components it is nonzero for $(10 - n) = p$ and for $(10 - n) = p + 2$. Performing the trace in each case one finds for the electric components

$$\begin{aligned} \text{Tr}(P_- \Gamma_{1(p)} M_p \gamma_\mu) &= \frac{16}{p!} \delta^{a_0}{}_\mu F_1^{a_1 \cdots a_p} \epsilon_{a_0 \cdots a_p} \\ \text{Tr}(P_- \Gamma_{1(p+2)} M_p \gamma_\mu) &= \frac{16}{(p+1)!} F_1^{a_0 \cdots a_p}{}_\mu \epsilon_{a_0 \cdots a_p} \end{aligned} \quad (12)$$

We have not pay attention to the signs on the right hand sides because we are interested in a_2 which is quadratic multiple of each term. One can easily verify that a_2 is nonzero only for the cases

$$\begin{aligned} a_2(n = m = p) &= \frac{8 \times 16}{n!} F_1^{(n)} \cdot V \cdot F_2^{(n)} \\ a_2(n = m = p + 2) &= \frac{8 \times 16}{(n-1)!} (F_1^{(n)})^{a_0 \cdots a_p}{}_i (F_2^{(n)})_{a_0 \cdots a_p}{}^i \end{aligned}$$

where the notation $F_1^{(n)} \cdot V \cdot F_2^{(n)}$ means the indices are contracted with the world volume metric η^{ab} .

For the magnetic components, one finds

$$\begin{aligned} a_2(n' = m' = p) &= \frac{8 \times 16}{n'!} (*F_1^{(n)}) \cdot V \cdot (*F_2^{(n)}) \\ a_2(n' = m' = p + 2) &= \frac{8 \times 16}{(n'-1)!} (*F_1^{(n)})^{a_0 \cdots a_p}{}_i (*F_2^{(n)})_{a_0 \cdots a_p}{}^i \end{aligned}$$

where $n' = 10 - n$. Again the kinematic factor a_2 for the magnetic components is the same as for the electric components but for $*F$.

The kinematic factor for the electric RR fields is

$$K(1, 2) = - \sum_n (2a_1 p_1 \cdot V \cdot p_2 + a_2 p_1 \cdot p_2 [\delta_{n,p} + \delta_{n,p+2}]) \quad (13)$$

where the summation is over $n = 1, 2, 3, 4, 5$. The field theory corresponding to this kinematic factor for D_p -brane in the string frame is

$$S_{DBI} \supset -\frac{\pi^2 \alpha'^2 T_p}{48 \times 32} \int d^{p+1}x e^\phi \sqrt{-G} \sum_n \frac{16}{n!} \left[(p-4) F^{(n)}_{,a} \cdot F^{(n),a} - n D^\mu{}_\nu F^{(n)}_{\mu,a} \cdot F^{(n)\nu,a} \right. \\ \left. + 4 F^{(n)}_{,\mu} \cdot V \cdot F^{(n),\mu} \delta_{n,p} + 4n F^{(n)}_{i,\mu} \cdot V \cdot F^{(n)i,\mu} \delta_{n,p+2} \right] \quad (14)$$

where we have also used the standard convention that the RR fields are rescaled as $C \rightarrow e^\phi C$. This is the reason why there is the dilaton factor e^ϕ in above action instead of the expected factor of $e^{-\phi}$ for the disk amplitude. Similar rescaling has been also used in couplings (5). The magnetic couplings are the same as above with replacing F s with $*F$ s. The above action gives all quadratic RR couplings at order $O(\alpha'^2)$ for any D_p -brane. However, we are interested in the world volume couplings of the self-dual D_3 -brane.

3 S-duality of DBI part

Let us start with the couplings of the two-forms. The couplings of NSNS two-form are given in (4) which have been found in [9] by consistency of the curvature couplings with T-duality. Compatibility of these couplings with S-duality requires similar couplings for the RR two-form. Writing the spacetime indices in (14) in terms of the world volume and the transverse indices, one finds that for D_3 -brane and for RR two-form the above action simplifies to

$$S_{DBI} \supset -\frac{\pi^2 \alpha'^2 T_3}{48} \int d^4x e^\phi \sqrt{-G} \left[\frac{1}{6} F_{ijk,a} F^{ijk,a} + \frac{1}{3} F_{abc,i} F^{abc,i} - \frac{1}{2} F_{bci,a} F^{bci,a} \right] \quad (15)$$

These couplings are similar to the B-field couplings in (4).

To study the transformation of the above couplings under S-duality, one should first rescale the metric from string frame to the Einstein frame $G_{\mu\nu} = e^{\phi/2} g_{\mu\nu}$. The B-field couplings (4) are multiplied by $e^{-2\phi}$ and the dilaton drops out of the above RR couplings. The D_3 -brane and the Einstein frame metric are invariant under $SL(2, Z)$, and the B-field and the RR two-form transform as doublet, *i.e.*,

$$\mathcal{B}^{(2)} = \begin{pmatrix} B^{(2)} \\ C^{(2)} \end{pmatrix} \quad (16)$$

Under a $SL(2, Z)$ transformation by

$$\Lambda = \begin{pmatrix} d & c \\ b & a \end{pmatrix} \quad (17)$$

the \mathcal{B} -field transforms linearly by the rule

$$\mathcal{B} \rightarrow \Lambda \mathcal{B} \quad (18)$$

The axion and the dilaton combine into a complex scalar field $\tau = C + ie^{-\phi}$. This field transforms as

$$\tau \rightarrow \frac{a\tau + b}{c\tau + d} \quad (19)$$

Now consider the matrix \mathcal{M}

$$\mathcal{M} = e^\phi \begin{pmatrix} |\tau|^2 & -C \\ -C & 1 \end{pmatrix} \quad (20)$$

which transforms under the $SL(2, Z)$ as

$$\mathcal{M} \rightarrow (\Lambda^{-1})^T \mathcal{M} \Lambda^{-1} \quad (21)$$

This matrix appears in the $SL(2, Z)$ form of the type IIB supergravity. Using this matrix, one can rewrite the couplings (4) and (15) in the Einstein frame as:

$$S_{DBI} \supset -\frac{\pi^2 \alpha'^2 T_3}{48} \int d^4x e^{-\phi} \sqrt{-g} \left[\frac{1}{6} \mathcal{F}_{ijk,a}^T \mathcal{M} \mathcal{F}^{ijk,a} + \frac{1}{3} \mathcal{F}_{abc,i}^T \mathcal{M} \mathcal{F}^{abc,i} - \frac{1}{2} \mathcal{F}_{bci,a}^T \mathcal{M} \mathcal{F}^{bci,a} \right] \quad (22)$$

where $\mathcal{F} = d\mathcal{B}$. Apart from the overall dilaton factor $e^{-\phi}$, it is invariant under the $SL(2, Z)$ transformation.

The above situation is like the R^4 corrections to the supergravity action which apart from the factor $\zeta(3)e^{-3\phi/2}$, the couplings are invariant under the $SL(2, Z)$ transformation. In that case, the tree level, one loop and one-instanton results are the three leading order terms of the non-holomorphic Eisenstein series E_s with $s = 3/2$. So it has been conjectured in [18] that the $SL(2, Z)$ invariant coupling is $E_{3/2}R^4$. In particular, this conjecture indicates that there is no perturbative corrections to R^4 other than one loop. It has been shown in [19] that there is no two loop correction to this action.

For general s , this series is defined by

$$2\zeta(2s)E_s(\tau, \bar{\tau}) = \sum_{(m,n) \neq (0,0)} \frac{\tau_2^s}{|m + n\tau|^{2s}} \quad (23)$$

where $\tau = \tau_1 + i\tau_2$. It is invariant under the $SL(2, Z)$ transformation. This function satisfies the following differential equation:

$$4\tau_2^2 \partial_\tau \partial_{\bar{\tau}} E_s = s(s-1)E_s \quad (24)$$

This equation has two solutions τ_2^s and τ_2^{1-s} corresponding to two particular orders of perturbation theory, and infinite number of non-perturbative solutions. For $s = 1$, however, the series (23) diverges logarithmically. The regularized function has the following expansion [6, 20]:

$$2\zeta(2)E_1(\tau, \bar{\tau}) = \zeta(2)\tau_2 - \frac{\pi}{2} \ln(\tau_2) + \pi\sqrt{\tau_2} \sum_{m \neq 0, n \neq 0} \left| \frac{m}{n} \right|^{1/2} K_{1/2}(2\pi|mn|\tau_2) e^{2\pi i mn \tau_1} \quad (25)$$

where the first term corresponds to $n = 0$ in the series (23). This term is exactly the dilaton factor in (22). This may indicate that the factor $e^{-\phi}$ in (22) should be replaced by the $SL(2, Z)$ invariant function E_1 .

Another evidence for this replacement is the following. The B-field couplings (4) are related to the gravity couplings (2) by T-duality [9]. On the other hand the $SL(2, Z)$ invariant form of the gravity couplings has this nonperturbative factor [6, 20]. In fact, the regularized function (25) is proportional to $\log(\tau_2 |\eta(\tau)|^4)$ [6] where $\eta(\tau)$ is the Dedekind η -function. The $\log \tau_2$ piece is nonanalytic and comes from the annulus [20] and the remaining part appears in the Wilsonian effective action found in [6]. Hence, one expects the $SL(2, Z)$ invariant form of the \mathcal{B} -field couplings (22) to be

$$S_{DBI} \supset -\frac{\pi^2 \alpha'^2 T_3}{24} \int d^4x E_1(\tau, \bar{\tau}) \sqrt{-g} \left[\frac{1}{6} \mathcal{F}_{ijk,a}^T \mathcal{M} \mathcal{F}^{ijk,a} + \frac{1}{3} \mathcal{F}_{abc,i}^T \mathcal{M} \mathcal{F}^{abc,i} - \frac{1}{2} \mathcal{F}_{bci,a}^T \mathcal{M} \mathcal{F}^{bci,a} \right] \quad (26)$$

The second term in the expansion of E_1 is the annulus contribution to the 1PI effective action. All other terms are D-instanton contributions.

The next case that we consider is the couplings of two $F^{(5)}$ on the world volume of D₃-brane. The RR potential $C^{(4)}$ is invariant under the $SL(2, Z)$ transformation. One can easily confirm that the couplings (14) for $C^{(4)}$ in the Einstein frame have the dilaton factor $e^{-\phi}$. Replacing this factor with the $SL(2, Z)$ invariant function $E_1(\tau, \bar{\tau})$, one finds the following S-dual couplings:

$$S_{DBI} \supset -\frac{\pi^2 \alpha'^2 T_3}{48 \times 5!} \int d^4x E_1(\tau, \bar{\tau}) \sqrt{-g} \left[-F^{(5)}_{,a} \cdot F^{(5),a} - 5 D^\mu{}_\nu F^{(5)}_{\mu,a} \cdot F^{(5)\nu,a} + 20 F^{(5)}_{i,\mu} \cdot V \cdot F^{(5)i,\mu} \right] \quad (27)$$

where we have also included the $*F^{(5)}$ terms. Similar couplings as those in the first line above, without the factor E_1 , can be written for the $F^{(5)}$ couplings on the world volume of D₇-brane. Since it is S-duality invariant, the $F^{(5)}$ couplings in (14) are the couplings for all (p, q) 7-branes.

The disk level S-matrix element of two NSNS vertex operators shows that there is no couplings for one dilaton and one graviton at order $O(\alpha'^2)$ for D₃-brane [9]. The kinematic factor for D₃-brane is

$$K(\phi, \varepsilon) = \text{Tr}(\varepsilon) \left(\frac{4q^4}{\sqrt{8}} \right) \quad (28)$$

where ε is the polarization of the symmetric tensor. Using the traceless of the graviton polarization, one finds the above kinematic factor is zero for graviton⁶. This is consistent

⁶This term which is zero for one graviton and one dilaton, did not considered in [9]. This term has non-zero contribution for the kinematic factor of two dilatons.

with S-duality because RR couplings (5) have no coupling between one graviton and one axion. It is also consistent with the couplings (3). Using various on-shell relations, one can show that in the Einstein frame these couplings become

$$S_{DBI} \supset \frac{\pi^2 \alpha'^2 T_p}{48} \int d^{p+1}x e^{-\phi} \sqrt{-g} \left[R_{abcd} R^{abcd} - 2\hat{R}_{ab} \hat{R}^{ab} - R_{abij} R^{abij} + 2\hat{R}_{ij} \hat{R}^{ij} \right. \\ \left. - (p-3)[\hat{R}_{ab} \phi^{,ab} - \hat{R}_{ij} \phi^{,ij}] - \frac{(p-3)^2}{8} [\phi_{,ab} \phi^{,ab} - \phi_{,ij} \phi^{,ij}] + \phi_{,ab} \phi^{,ab} \right] \quad (29)$$

which gives zero coupling for the linear dilaton for the case $p = 3$. Replacing the polarization tensor ε in (28) with the dilaton polarization, one finds the last term in the above equation.

Under the S-duality both the dilaton and the RR scalar appears in the complex field τ , so one expects that there should be similar coupling for the RR scalar at order $O(\alpha'^2)$ for D₃-brane. The couplings (14) for the RR scalar for the D₃-brane in the Einstein frame become

$$S_{DBI} \supset \frac{\pi^2 \alpha'^2 T_3}{48} \int d^4x e^{\phi} \sqrt{-g} C_{,ab} C^{,ab} \quad (30)$$

To study the S-duality of this coupling we add the dilaton coupling in (29) to the above equation, *i.e.*,

$$S_{DBI} \supset \frac{\pi^2 \alpha'^2 T_3}{48} \int d^4x e^{-\phi} \sqrt{-g} \left[\phi_{,ab} \phi^{,ab} + e^{2\phi} C_{,ab} C^{,ab} \right] \quad (31)$$

The terms in the bracket can be extended to the $SL(2, Z)$ invariant form by including the disk-level nonlinear terms, and the dilaton factor can be extended to the $SL(2, Z)$ invariant form by including the annulus and the D-instanton effects. The S-dual extension of the coupling (31) is

$$S_{DBI} \supset -\frac{\pi^2 \alpha'^2 T_3}{96} \int d^4x E_1(\tau, \bar{\tau}) \sqrt{-g} \text{Tr}(\mathcal{M}_{,ab} (\mathcal{M}^{-1})^{,ab}) \quad (32)$$

where the matrix \mathcal{M} is the one appears in (20)

4 S-duality of CS part

The linear NSNS couplings in the CS action are given in (5). To study the S-duality of these couplings for the self-dual D₃-brane, we begin by witting the couplings in the first line in the Einstein frame, *i.e.*,

$$S_{CS} \supset -\frac{\pi^2 \alpha'^2 T_3}{24 \times 2!2!} \int d^4x \epsilon^{a_0 \dots a_3} e^{-\phi} \left[F_{ia_2 a_3, a} H_{a_0 a_1}{}^{a, i} - F_{aa_2 a_3, i} H_{a_0 a_1}{}^{i, a} \right] \quad (33)$$

To write it in a $SL(2, Z)$ invariant form, we first introduce the $SL(2, Z)$ matrix

$$\mathcal{N} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad (34)$$

This matrix has the property

$$\mathcal{N} = (\Lambda^{-1})^T \mathcal{N} \Lambda^{-1} \quad (35)$$

Using this matrix, one can rewrite the terms in the parenthesis in the S-dual form $\mathcal{F}^T \mathcal{N} \mathcal{F}$. Then using the same idea as in previous section, one can extend it to the following $SL(2, Z)$ invariant form:

$$S_{CS} \supset -\frac{\pi^2 \alpha'^2 T_3}{12 \times 2!2!} \int d^4 x \epsilon^{a_0 \dots a_3} E_1(\tau, \bar{\tau}) \mathcal{F}^T_{a_0 a_1 a, i} \mathcal{N} \mathcal{F}_{a_2 a_3}{}^{i, a} \quad (36)$$

The second term in the expansion of $E_1(\tau, \bar{\tau})$ can be calculated with the annulus level S-matrix element of one RR and one NSNS vertex operators.

The RR couplings in the second line of (5) for the D₃-brane in the Einstein frame are proportional to the dilaton factor $e^{-\phi}$. In particular, the term $\hat{\mathcal{R}}^{ij} - \phi^{,ij}$ becomes proportional to $\hat{\mathcal{R}}^{ij} + \eta^{ij} \phi_{,a}{}^a / 4$ in the Einstein frame. The last term gives $F_{a_0 \dots a_3 i}^{(5), i}$ which is zero on-shell, *i.e.*, $F_{a_0 \dots a_3 i}^{(5), i} = -F_{a_0 \dots a_3 a}^{(5), a}$ where two world volume indices in $F_{a_0 \dots a_3 a}^{(5)}$ is identical, hence, it is zero. Using the fact that the Einstein frame metric and $C^{(4)}$ are invariant under the S-duality, and replacing $e^{-\phi}$ with $E_1(\tau, \bar{\tau})$, one finds the following S-dual couplings:

$$S_{CS} \supset -\frac{\pi^2 \alpha'^2 T_3}{6} \int d^4 x \epsilon^{a_0 \dots a_3} E_1(\tau, \bar{\tau}) \left[\frac{1}{2!3!} F_{ia_1 \dots a_3 j, a}^{(5)} \mathcal{R}^a{}_{a_0}{}^{ij} - \frac{1}{4!} F_{a_0 \dots a_3 j, i}^{(5)} \hat{\mathcal{R}}^{ij} \right]$$

Note that the above couplings are not invariant under the standard form of T-duality transformations. This is because the standard T-duality rules are in string frame whereas the above S-dual couplings are in the Einstein frame. To study the T-duality of the above couplings one should first write the T-duality transformations in the Einstein frame and then applied to the above action.

The coupling in the third line of (5) for D₃-brane is

$$S_{CS} \supset \frac{\pi^2 \alpha'^2 T_3}{24} \int d^4 x \epsilon^{a_0 \dots a_3} \left(\frac{1}{3!4!} F_{ia_0 \dots a_3 j k, a}^{(7)} H^{ijk, a} \right) \quad (37)$$

Using the fact that the redundant RR field strength $F^{(7)}$ is the magnetic dual of $F^{(3)}$, *i.e.*, $F^{(7)} = *F^{(3)}$, and the relation

$$(*F^{(n)})_{\mu_{n+1} \dots \mu_{10}} = \frac{1}{n!} \sqrt{-G} \epsilon_{\mu_1 \dots \mu_{10}} (F^{(n)})^{\mu_1 \dots \mu_n} \quad (38)$$

one finds that in the Einstein frame again the coupling (37) has the dilaton factor $e^{-\phi}$. Moreover, the coupling is antisymmetric under changing the RR two form with the B-field. So the coupling (37) can be written in the Einstein frame as

$$S_{CS} \supset \frac{\pi^2 \alpha'^2 T_3}{24 \times 2} \int d^4 x \epsilon^{a_0 \dots a_3} e^{-\phi} \left(\frac{1}{3!4!} (*F^{(3)})_{ia_0 \dots a_3 jk, a} H^{ijk, a} - \frac{1}{3!4!} (*H)_{ia_0 \dots a_3 jk, a} (F^{(3)})^{ijk, a} \right)$$

Defining the magnetic dual of $\mathcal{F}^{(3)} = d\mathcal{B}^{(2)}$ as

$$\mathcal{F}^{(7)} = *\mathcal{F}^{(3)} = \begin{pmatrix} *H^{(3)} \\ *F^{(3)} \end{pmatrix}, \quad (39)$$

one finds the following S-dual coupling:

$$S_{CS} \supset -\frac{\pi^2 \alpha'^2 T_3}{24 \times 3!4!} \int d^4 x \epsilon^{a_0 \dots a_3} E_1(\tau, \bar{\tau}) (\mathcal{F}^{(7)})_{a_0 \dots a_3 ijk, a}^T \mathcal{N} \mathcal{F}^{ijk, a} \quad (40)$$

Similar coupling as above, without the factor E_1 , can be written for the couplings in the first line of (5) for D7-brane. Therefore, the couplings in the first line of (5) are the couplings for all (p, q) 7-branes.

Finally we consider the Chern-Simons couplings. Compatibility of the CS action (1) with T-duality requires the couplings (6). We now try to write these B-field couplings in S-dual invariant form. The couplings (6) for the self-dual D3-brane are

$$S_{CS} \supset -\frac{\pi^2 \alpha'^2 T_3}{48 \times 2!2!} \int d^4 x \epsilon^{a_0 \dots a_3} C \left[H_{a_0 a_1 a, i} H_{a_2 a_3}{}^{a, i} - H_{a_0 a_1 i, a} H_{a_2 a_3}{}^{i, a} \right] \quad (41)$$

The above couplings have been also confirmed with scattering calculation [10, 12]. The compatibility of these couplings with S-duality indicates that the disk level S-matrix element of three RR vertex operators produces the following couplings:

$$S_{CS} \supset -\frac{\pi^2 \alpha'^2 T_3}{48 \times 2!2!} \int d^4 x \epsilon^{a_0 \dots a_3} e^{2\phi} C \left[F_{a_0 a_1 a, i} F_{a_2 a_3}{}^{a, i} - F_{a_0 a_1 i, a} F_{a_2 a_3}{}^{i, a} \right] \quad (42)$$

where we have also used the standard rescaling $C \rightarrow e^\phi C$. In the Einstein frame the terms in the brackets can be combined into the $SL(2, Z)$ invariant form of $\mathcal{F}^T \mathcal{M} \mathcal{F}$. However, in this case amplitude has no overall dilaton factor $e^{-\phi}$. Instead, it has the RR scalar as the overall factor. The S-dual invariant of this coupling then does not include the non-holomorphic Eisenstein function. It should include another modular function $f(\tau, \bar{\tau})$, which has the following weak-expansion:

$$f(\tau, \bar{\tau}) \sim \tau_1 + \dots \quad (43)$$

where dots stands for loops and D-instanton effects.

It has been shown in [6] that the gravity couplings in the effective Wilsonian CS theory for trivial normal bundle can be written in S-dual invariant form. The modular function

that appears in this case is $\log(\eta(\tau)/\eta(\bar{\tau}))$. This function has the following weak-expansion [6]:

$$\log \frac{\eta(\tau)}{\eta(\bar{\tau})} = \frac{i\pi}{6}\tau_1 - \left[q + \frac{3}{2}q^2 + \frac{4}{3}q^3 + \cdots - cc \right] \quad (44)$$

where $q = e^{2\pi i\tau}$. The first term arises from the disk level amplitude and the series of the power q stand for the D-instanton corrections. The annulus effect is absent in the above function as it does not contribute to the Wilsonian effective action. On the other hand the function $\log(\eta(\tau)/\eta(\bar{\tau}))$ is not invariant under the $SL(2, Z)$ transformation. The variation of the Wilsonian effective action cancels the anomalous contributions of the massless modes of D₃-brane [6]. The annulus effect in the 1PI effective action should make this function to be $SL(2, Z)$ invariant. So we expect the $SL(2, Z)$ invariant function $f(\tau, \bar{\tau})$ to be

$$f(\tau, \bar{\tau}) = \log \frac{A(\tau)\eta(\tau)}{A(\bar{\tau})\eta(\bar{\tau})} \quad (45)$$

where $A(\tau)$ is the annulus effect which makes $f(\tau, \bar{\tau})$ to be $SL(2, Z)$ invariant. In terms of this function, one can write the above couplings in the following S-dual form:

$$S_{CS} \supset \frac{i\pi\alpha'^2 T_3}{8 \times 2!2!} \int d^4x \epsilon^{a_0 \cdots a_3} f(\tau, \bar{\tau}) \left[\mathcal{F}_{a_0 a_1 a, i}^T \mathcal{M} \mathcal{F}_{a_2 a_3}{}^{a, i} - \mathcal{F}_{a_0 a_1 i, a}^T \mathcal{M} \mathcal{F}_{a_2 a_3}{}^{i, a} \right] \quad (46)$$

It would be interesting to perform the annulus calculation to find the function $A(\tau)$.

For D₇-brane, the couplings are the same as above with replacing $f(\tau, \bar{\tau})$ with the $SL(2, Z)$ invariant field $C^{(4)}$. The couplings (41) have been verified by the S-matrix element of one RR and two NSNS vertex operators [10, 12]. The S-matrix element produces some other couplings as well [12]. Using the same steps as for (41), one can write them in $SL(2, Z)$ invariant forms. This ends our illustration of the consistency of the D-brane action at order $O(\alpha'^2)$ with S-duality. The consistency of the DBI action (2) with S-duality predicts the following couplings in the Einstein frame at the disk level:

$$\begin{aligned} S_{DBI} &\supset -\frac{\pi^2 \alpha'^2 T_3}{48} \int d^4x \sqrt{-g} C^2 \left[\frac{1}{6} H_{ijk, a} H^{ijk, a} + \frac{1}{3} H_{abc, i} H^{abc, i} - \frac{1}{2} H_{bci, a} H^{bci, a} \right] \\ S_{DBI} &\supset \frac{\pi^2 \alpha'^2 T_3}{24} \int d^4x \sqrt{-g} C \left[\frac{1}{6} H_{ijk, a} F^{ijk, a} + \frac{1}{3} H_{abc, i} F^{abc, i} - \frac{1}{2} H_{bci, a} F^{bci, a} \right] \end{aligned} \quad (47)$$

The consistency of the CS action (1) with S-duality predicts the couplings (42).

$$\begin{aligned} S_{CS} &\supset -\frac{\pi^2 \alpha'^2 T_3}{48 \times 2!2!} \int d^4x \epsilon^{a_0 \cdots a_3} e^\phi C^3 \left[H_{a_0 a_1 a, i} H_{a_2 a_3}{}^{a, i} - H_{a_0 a_1 i, a} H_{a_2 a_3}{}^{i, a} \right] \\ S_{CS} &\supset \frac{\pi^2 \alpha'^2 T_3}{24 \times 2!2!} \int d^4x \epsilon^{a_0 \cdots a_3} e^\phi C^2 \left[H_{a_0 a_1 a, i} F_{a_2 a_3}{}^{a, i} - H_{a_0 a_1 i, a} F_{a_2 a_3}{}^{i, a} \right] \\ S_{CS} &\supset -\frac{\pi^2 \alpha'^2 T_3}{48 \times 2!2!} \int d^4x \epsilon^{a_0 \cdots a_3} e^\phi C \left[F_{a_0 a_1 a, i} F_{a_2 a_3}{}^{a, i} - F_{a_0 a_1 i, a} F_{a_2 a_3}{}^{i, a} \right] \end{aligned} \quad (48)$$

Moreover, since the modular functions $E_1(\tau, \bar{\tau})$ and $f(\tau, \bar{\tau})$ have annulus contributions, the S-duality also predicts the annulus level couplings for the S-duality invariant couplings that we have found, *e.g.*, (26). It would be interesting to confirm these couplings by direct calculations.

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